

## Quantum nonlocality, Bell inequalities, and the memory loophole

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In the analysis of experiments designed to reveal violation of Bell-type inequalities, it is usually assumed that any hidden variables associated with the  $n$ th particle pair would be independent of measurement choices and outcomes for the first  $(n-1)$  pairs. Models which violate this assumption exploit what we call the *memory loophole*. We focus on the strongest type of violation, which uses the *two-sided* memory loophole, in which the hidden variables for pair  $n$  can depend on the previous measurement choices and outcomes in both wings of the experiment. We show that the two-sided memory loophole allows a systematic violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality when the data are analyzed in the standard way, but cannot produce a violation if a CHSH expression depending linearly on the data is used. In the first case, the maximal CHSH violation becomes small as the number of particle pairs tested becomes large. Hence, although in principle the memory loophole implies a slight flaw in the existing analyses of Bell experiments, the data still strongly confirm quantum mechanics against local hidden variables. We consider also a related loophole, the *simultaneous measurement loophole*, which applies if all measurements on each side are carried out simultaneously. We show that this can increase the probability of violating the linearized CHSH inequality as well as other Bell-type inequalities.

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### I. THE MEMORY LOOPHOLE

Bell's work [1] in the early 1960s made precise the sense in which classical intuitions based on the principles of special relativity conflict with quantum theory. Theoretical and experimental investigations have continued ever since, leading, *inter alia*, to the understanding of entanglement as a quantifiable resource of fundamental importance for quantum cryptography, communication, and computation.

The experiment analyzed by Bell is the following [1]. A source prepares a pair of particles in some entangled state. One particle is sent to Alice and one to Bob, Alice and Bob being situated far from each other. When the particle arrives at Alice, Alice subjects it to a measurement  $X$ , chosen by her at random amongst many possible measurements  $A_1, A_2$ , etc. Similarly, Bob subjects his particle to a measurement  $Y$  selected by him at random amongst many possible measurements  $B_1, B_2$ , etc. The experiment is repeated many times. Everything is arranged such that each pair of measurements performed by Alice and Bob is spacelike separated. After the experiment ends, Alice and Bob come together and compare their results.

Bell asked whether the correlations between the results of the measurements predicted by quantum mechanics can be explained by any classical model. More precisely, he formulated a model, known as a *local hidden variable model*, which is supposed to describe *all possible ways* in which classical systems can generate correlated answers in an ex-

periment as above. He then went on to prove that quantum-mechanical correlations cannot be obtained from such a model.

The key words above are *all possible ways*. To guarantee that one has found all possible ways in which a system may behave is problematic, and formally not a well-defined statement. Nevertheless, Bell's model, which we describe in detail below, is very powerful, and it has been generally accepted that it covers indeed all possibilities. Here, however, we argue that there are possibilities that have not been accounted for in Bell's model, which rely in one way or another on what, for reasons that will be obvious, we call the *memory loophole*.

The rest of our paper is organized as follows. In this section we describe Bell's original hidden variable model and present the memory loophole. In Sec. II, we introduce an inequality that is equivalent to the Clauser-Horne-Shimony-Holt (CHSH) inequality [2] and define some terms. In Sec. III, we summarize the results of the paper. In Secs. IV to VI, we analyze the inequality from the point of view of Bell's original model and various different versions of the memory loophole. We show that the probability of violating a standard CHSH inequality is affected by the loophole, but that the effect is not significant for a large sample. Finally, in Sec. VII, we consider a related loophole which arises in experiments in which all  $N$  measurements on each side are made simultaneously. Section VIII concludes.

The model proposed by Bell is the following. When two particles are prepared at the source in some (entangled) state  $\Psi$ , they both receive an index  $\lambda$  which is called a *local hidden variable*. This index is chosen at random according to some distribution  $\rho(\lambda)$ . The hidden variable essentially prescribes, in a *local* way, how the particles behave when sub-

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jected to different measurements. That is, when Alice subjects her particle to a measurement  $A$ , the particle gives an outcome  $a$  according to some probability distribution  $P(a;A,\lambda)$ , which depends on the measurement  $A$  and on the hidden variable  $\lambda$  but not on the measurement  $B$  performed by Bob on his particle or on the result  $b$  of this measurement. Similarly, Bob's particle yields an outcome  $b$  according to the probability distribution  $P(b;B,\lambda)$ , which depends on the measurement  $B$  to which it is subjected and on the hidden variable  $\lambda$  but not on  $A$  or  $a$ . The joint probability  $P(a,b;A,B)$  that the particles yield the outcomes  $a$  and  $b$  when subjected to the measurement of  $A$  and  $B$ , respectively, is then given by

$$P(a,b;A,B) = \int d\lambda \rho(\lambda) P(a;A,\lambda) P(b;B,\lambda). \quad (1)$$

The above model has been hitherto considered to describe all possible ways in which classical particles can yield long distance correlations while respecting the relativistic constraint of no superluminal signaling, which prevents Alice's particle from modifying its behavior according to what Bob does if there is not enough time for a light signal to arrive from Bob to Alice, and vice versa. Bell showed that quantum mechanics predicts correlations which cannot be obtained from such a model. The inconsistency of quantum theory with the hypothesis of local hidden variables is often—slightly confusingly—referred to as quantum nonlocality.

A way of testing whether or not some given correlations can be obtained from a local hidden variable (LHV) model is to test some signatures of such models, called Bell inequalities. The best known Bell-type inequality is the CHSH inequality. Suppose that Alice and Bob chose at random between two measurements  $A_1$  or  $A_2$  and  $B_1$  or  $B_2$ , respectively. Suppose, furthermore, that each of these measurements has only two possible outcomes,  $+1$  and  $-1$ . Then as CHSH have shown [2], if the particles behave according to any LHV model,

$$E(A_1B_1) + E(A_2B_1) + E(A_1B_2) - E(A_2B_2) \leq 2, \quad (2)$$

where  $E(AB)$  denotes the expectation value of the product of the outcomes of the measurements  $A$  and  $B$ . On the other hand, one can find quantum-mechanical states  $|\Psi\rangle$  (for example, any entangled pure state [3,4]) and appropriate measurements so that the CHSH inequality is violated. For example, if the state  $|\Psi\rangle$  is the singlet state of two spin- $(1/2)$  particles,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|+1\rangle|-1\rangle - |-1\rangle|+1\rangle), \quad (3)$$

where  $|+1\rangle$  and  $|-1\rangle$  represent the spin polarized “up” and “down” along the  $z$  axis, one can find appropriate spin measurements which yield

$$\langle\Psi|A_1B_1|\Psi\rangle + \langle\Psi|A_2B_1|\Psi\rangle + \langle\Psi|A_1B_2|\Psi\rangle - \langle\Psi|A_2B_2|\Psi\rangle = 2\sqrt{2}, \quad (4)$$

violating the LHV limit (2). (Here we have used the quantum-mechanical formula for the expectation value  $E(AB) = \langle\Psi|AB|\Psi\rangle$ .)

Bell's LHV model has generally been thought to cover all possible ways in which classical particles can behave. We show now, however, that this is not the case.

In order to determine correlations one has to perform measurements not on a single pair of particles but on many such pairs, and gather a large number of outcomes which will determine the statistics. Now, according to the LHV model above, Eq. (1), all the pairs in the ensemble are *uncorrelated*. This assumption appears natural from the perspective of quantum mechanics. In quantum theory, when we have a number of pairs, each pair being described by the same wave function, the pairs are uncorrelated. However, we can imagine the following scenario. A first pair of particles is emitted by the source. One of the particles arrives at Alice and it is subjected to a measurement and gives an outcome according to the LHV model (1). However, it also leaves in the environment information indicating to what measurement it was subjected and what outcome it yielded. Now, when a particle in the second pair arrives at Alice, it will read this message and it will give an outcome that depends not only on the measurement it is subjected to, but also on the message left by the first particle, i.e., on what has happened to the first particle. Particles on Bob's side behave in a similar way. The consequence is that the original LHV model (1) is now replaced by

$$P(a^{(n)}, b^{(n)} | A^{(n)}, B^{(n)}) = \int d\lambda \rho(\lambda) P(a^{(n)} | A^{(n)}, M, \lambda) P(b^{(n)} | B^{(n)}, M, \lambda), \quad (5)$$

where

$$P(a^{(n)} | A^{(n)}, M, \lambda) = P(a^{(n)} | A^{(n)}, A^{(1)}, \dots, A^{(n-1)}, a^{(1)}, \dots, a^{(n-1)}, \lambda) \quad (6)$$

and

$$P(b^{(n)} | B^{(n)}, M, \lambda) = P(b^{(n)} | B^{(n)}, B^{(1)}, \dots, B^{(n-1)}, b^{(1)}, \dots, b^{(n-1)}, \lambda). \quad (7)$$

Here  $M$  stands for the local record, or *memory*, of the previous measurements. We call this a local hidden variable model with *one-sided memory*.

There is a further interesting variation of Bell's original model. Suppose that the source emits pairs of correlated particles one by one. Suppose too that on each pair Alice and Bob perform their measurements spacelike separated from one another, so while Alice is performing her measurement, no signal can arrive from Bob's measurement. However, the time between the measurements on the different pairs is long enough, so that by the time Alice measures her  $n$ th particle, the particle could have received information about what has

happened in Bob’s measurements on all previous particles  $(1, \dots, n-1)$ , and similarly for Bob. One could imagine local hidden variable models in which this information is indeed communicated and used, in which case the probability in Eq. (6) is replaced by

$$P(a^{(n)} | A^{(n)}, M, \lambda) = P(a^{(n)} | A^{(n)}, A^{(1)}, \dots, A^{(n-1)}, a^{(1)}, \dots, a^{(n-1)}, B^{(1)}, \dots, B^{(n-1)}, b^{(1)}, \dots, b^{(n-1)}, \lambda), \quad (8)$$

and similarly for the probability on Bob’s side. This is a local hidden variable model with *two-sided memory*.

In principle, Bell’s original argument can be extended to render both types of memory loopholes irrelevant. We could require that separated apparatuses are used for each particle pair, and that *every* measurement is spacelike separated from every other—but it seems unlikely that such an experiment will be done any time soon with a large enough sample of particles to demonstrate statistically significant violations of Bell inequalities. Even the much weaker constraint that all of Alice’s measurements are spacelike separated from all of Bob’s—which would exclude the two-sided but not the one-sided loophole—has not been satisfied in any experiment to date. (See, e.g., Ref. [9] and references therein).

It is worth emphasizing that the memory loopholes described above have a different status from that of other loopholes such as the well-known detection loophole [5] or the recently discussed collapse locality loophole [6]. The detection loophole does not identify a problem with Bell’s local hidden variable model *per se*, but only states that technological limitations have to be taken into account—which can be done in the framework of the original model. Similarly, given a precise theory of state reduction (which is required to characterize the loophole precisely in the first place), the collapse locality loophole could be closed by carrying out standard Bell experiments using sufficiently advanced technology [6]. On the other hand, although, as we have noted, the memory loophole could be eliminated by new types of Bell experiments, it does highlight an intrinsic limitation of Bell’s model as applied to standard Bell experiments. This is not to say that local hidden variable theories exploiting the memory loophole are necessarily more plausible than theories exploiting other loopholes (indeed, a contrary view can be argued [6]). It does, though, mean that there is need for a reanalysis of the power of general local hidden variable theories in standard Bell experiments.

Another interesting theoretical question arises if Alice decides to measure all her particles simultaneously, and Bob does likewise. Since the measurements take some finite time, all the particles on Alice’s side could conceivably communicate with each other. Hence the outcome, given any particle  $n$ , may depend on what happened with all the other particles (i.e., to what measurements they are subjected and what outcomes are they yielding). We call the resulting loophole the *simultaneous measurement loophole*, and the LHV models which exploit it *collective* LHV models.

One might wonder what the point of considering all these loopholes is. Each seems to involve more conspiracy on Na-

ture’s part than the last, and none of them appears to lead to plausible physical models. Given the importance of the Bell-type experiments, however, and their consequences for our world view, we feel that it is important to analyze the experiments as rigorously as possible and, in particular, to distinguish between the logical impossibility and physical implausibility of the models.

There is another more practical motivation [7,8]. It is well known that quantum key distribution schemes which use entanglement have significant security advantages over other schemes; they can also be extended by the use of quantum repeaters to allow secure key distribution over arbitrary distances. The security of these schemes relies crucially on the fact that the states created and measured are genuinely entangled. The most obvious and seemingly reliable way to verify this is to use Bell-type tests as security checks within the protocols. However, any such tests need to be interpreted with care. If a quantum cryptosystem is acquired from a not necessarily reliable source, or possibly exposed to sabotage, a cautious user must consider the possibility that devices have been installed which use classical communication to simulate, as far as possible, the behavior of quantum states, while allowing third parties to extract illicit information about the key. Such devices effectively define a local hidden variable model, and the usual criterion of physical plausibility no longer applies. A saboteur could set up communication and computing devices that use any information available anywhere in the cryptosystem. In particular, saboteurs might well try to exploit memory loopholes, as well as other Bell experiment loopholes, if they could gain a significant advantage by so doing.

Having established, therefore, that the original version of the local hidden variable model as proposed by Bell has to be modified, we now examine the consequences.

## II. CHSH-TYPE INEQUALITIES: GENERAL CONSIDERATIONS

We first revisit the usual Bell inequalities experiments and emphasize in more detail the statistical aspects of the measurements.

The standard CHSH inequality is described in Eq. (2) and it is claimed that every ordinary (i.e., as originally constructed by Bell) local hidden variable model must obey the inequality.

Of course, even in an ideal experiment, an ordinary local hidden variable model can violate the CHSH bound. The quantities which figure in the CHSH expression are theoretical expectation values, which are abstract concepts. In reality each expectation value is determined by repeating a measurement a large number of times and estimating the probabilities (and hence the expectation values) as frequencies of events. These measured expectation values are subject to statistical fluctuations, which can yield violations of the CHSH bound. Our first task is to examine the problem in detail, defining precisely the operational meaning of the different quantities, and get an accurate understanding of what exactly is the meaning of the violation of Bell’s inequalities. Only after all these are clarified will we be able to see the

effect of the various memory loopholes. In particular, we will see that memory can allow particles to take advantage of statistical fluctuations and build them up into a systematic bias. We will also see, however, that if the CHSH expressions are defined in the usual way, the biases that can thus be obtained tend to zero as the number of pairs tested increases. Moreover, we will see that a simpler linearized form of the CHSH expressions is “memory proof,” in the sense that the probability of a given level of violation is no greater for memory-dependent local hidden variable models than for optimally chosen memoryless models.

We use the CHSH inequality in the form [10]

$$P_{CHSH} = P_c(A_1, B_1) + P_c(A_1, B_2) + P_c(A_2, B_1) + P_a(A_2, B_2) \leq 3 \quad (9)$$

for local hidden variable theories, where  $P_c(A, B)$  is the probability that  $A$  and  $B$  have the same outcome (are correlated), and  $P_a(A, B)$  is the probability that  $A$  and  $B$  have different outcomes (are anticorrelated).  $A_1, A_2, B_1, B_2$  are chosen so that quantum mechanics predicts the maximal value,  $P_{CHSH} = 2 + \sqrt{2}$ .

What we actually mean by Eq. (9) in an experimental context is the following. We suppose that Alice and Bob perform measurements on  $N$  pairs of particles. For each of their particles Alice and Bob choose at random what measurement to perform,  $A$  or  $A'$  for Alice and  $B$  or  $B'$  for Bob. We define  $\mathfrak{N}(A, B)$  to be the number of pairs on which operators  $A$  and  $B$  were measured,  $\mathfrak{N}_c(A, B)$  and  $\mathfrak{N}_a(A, B)$  to be the number of times the outcomes were correlated and anticorrelated in these measurements. Note that Alice and Bob should not prearrange the sequence of their measurements—this would introduce well-known loopholes; the entire experiments of Alice and Bob, including the decision of what to measure on each particle, have to be spacelike separated from each other. Consequently Alice and Bob do not have total control on how many times a specific pair of measurements, say,  $A, B$  is performed, but this number,  $\mathfrak{N}(A, B)$  is a random variable.

We define

$$X_N = \frac{\mathfrak{N}_c(A_1, B_1)}{\mathfrak{N}(A_1, B_1)} + \frac{\mathfrak{N}_c(A_1, B_2)}{\mathfrak{N}(A_1, B_2)} + \frac{\mathfrak{N}_c(A_2, B_1)}{\mathfrak{N}(A_2, B_1)} + \frac{\mathfrak{N}_a(A_2, B_2)}{\mathfrak{N}(A_2, B_2)}, \quad (10)$$

$$Y_N = \frac{4}{N} [\mathfrak{N}_c(A_1, B_1) + \mathfrak{N}_c(A_1, B_2) + \mathfrak{N}_c(A_2, B_1) + \mathfrak{N}_a(A_2, B_2)]. \quad (11)$$

$X_N$  is the experimental meaning of the CHSH inequality (9); the index  $N$  denotes that the experiment has been performed on  $N$  pairs. Indeed, the expression  $\mathfrak{N}_c(A_1, B_1)/\mathfrak{N}(A_1, B_1)$  is the frequency of correlations between the outcomes of  $A_1$  and  $B_1$ , and it is therefore the experimental definition of the correlation probability  $P_c(A_1, B_1)$  and so on.

Note that our definition of  $X_N$  assumes that  $\mathfrak{N}(A, B) > 0$  for all pairs of operators  $A, B$ . If not,  $X_N$  is undefined. Strictly speaking, our expressions for the expectation and other functions of  $X_N$  should thus all be conditioned on the event that  $X_N$  is defined. We will neglect this below, assuming that  $N$  is large enough, so that the probability of  $X_N$  being undefined is negligible. One could, alternatively, use an experimental protocol which ensures that  $X_N$  is defined. For instance, one could require that, if  $\mathfrak{N}(A, B) = 0$  for any  $A, B$  after  $N$  pairs have been tested, the experiment continues on further pairs until  $\mathfrak{N}(A, B) > 0$  for all  $A, B$ , and then terminates. Our analyses would need to be modified slightly to apply to such a protocol, but the results would be essentially the same.

$Y_N$  is another experimental quantity closely related to  $X_N$ . The two quantities are equal if the expressions  $\mathfrak{N}(A_i, B_j)$  are equal for all  $i, j$ . For large  $N$ , the  $\mathfrak{N}(A_i, B_j)$  are almost always nearly equal, and so the same is true of  $X_N$  and  $Y_N$ . Although it is traditional to use  $X_N$  in analyzing Bell experiments,  $Y_N$  is in fact much better behaved and easier to analyze, since it is a linear expression.

### III. CHSH-TYPE INEQUALITIES: EXPECTATION VALUES AND FLUCTUATIONS

$X_N$  and  $Y_N$  represent quantities determined by making measurements on a batch of  $N$  pairs of particles. We do not assume that the pairs behave independently: they may be influenced by memory, and we will analyze the different types of memories. We are interested in the maximum possible expectation value of  $X_N$  and  $Y_N$ , and the maximum probability of  $X_N$  or  $Y_N$  taking a value much larger than the expectation.

Obviously, the expectation and fluctuations of  $X_N$  and  $Y_N$  could be experimentally estimated only by repeating the whole series of  $N$  experiments a large number of times, and then only under the assumption that different batches of  $N$  pairs behave independently. Without some restriction on the scope of the memory loophole, we would need to allow for the possibility that any experiment we perform in the future could, in principle, be influenced by the results obtained in all experiments to date.

Those who require probabilities to have a frequency interpretation in order to be meaningful may thus have some difficulty interpreting the results of memory loophole analyses. The only certain way to circumvent this difficulty would be to set up many spacelike separated experiments. On the other hand, if probability is viewed simply as a measure of the plausibility of a theory, there is no interpretational difficulty. As we will see, it can be shown that the probability of obtaining experimental data consistent with quantum theory, given a local hidden variable theory using a memory loophole, for a large sample, is extremely small. Since the cumulative data in Bell experiments are indeed consistent with quantum theory, we conclude that they effectively refute the hypothesis of memory-dependent local hidden variables—so long, of course, as these hidden variables are assumed not also to exploit other well-known loopholes such as the detector efficiency loophole.

The results for which we have complete proofs can be summarized in the following table:

LHV Model	$E(X_N)$	$P(\hat{X}_N > 5\delta)$	$E(Y_N)$	$P(\hat{Y}_N > \delta)$
Memoryless	$\leq 3$	$< 5f_N^\delta$	$\leq 3$	$< f_N^\delta$
1-sided Memory	$< 3 + o(N^{-1/2+\epsilon})$	$< 5f_N^\delta$	$\leq 3$	$< f_N^\delta$
Collective	?	?	$\leq 3$	?
2-sided Memory	$< 3 + o(N^{-1/2+\epsilon})$	$< 5f_N^\delta$	$\leq 3$	$< f_N^\delta$

Here  $\hat{X}_N = X_N - 3$ ,  $\hat{Y}_N = Y_N - 3$ , and we have simplified the presentation by taking  $\delta$  to be small enough that  $(3 + \delta) < (3 + 5\delta)(1 - \delta)$ . The expression  $o(N^{-1/2+\epsilon})$  denotes a term that asymptotically tends to zero faster than  $N^{-1/2+\epsilon}$  for any  $\epsilon > 0$ , and

$$f_N^\delta = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{3}}{\delta\sqrt{N}} \exp\left(-\frac{1}{6}\delta^2 N\right). \quad (12)$$

The proofs are given in the following sections.

The significance of these results is as follows. The memoryless case represents the results for standard local hidden variables which behave independently for each pair. The result  $E(X_N) \leq 3$  is the standard expression of the CHSH inequality. Although values of  $X_N$  larger than 3 can be experimentally obtained from a local hidden variable model, the probability of obtaining  $3 + \delta$  decreases exponentially as  $5f_N^\delta$ . Hence, for a given  $\delta$  and sufficiently large  $N$ , observing  $3 + \delta$  when performing  $N$  experiments can be taken as a very good confirmation of the fact that it is not due to an LHV model. In the memoryless case,  $E(Y_N) \leq 3$  and the fluctuations also decrease exponentially.

In the two-sided memory case, the expectation value of  $Y_N$  again satisfies  $E(Y_N) \leq 3$ . Hence the existence of memory makes no difference here. Memory also makes no difference to the fluctuations: they still decrease exponentially. On the other hand, the expectation value of  $X_N$  can be larger than in the standard memoryless case. Hypothetically, if Bell experiments are analyzed by using  $X_N$  and the effect of the memory loophole is neglected, a two-sided memory LHV model could mistakenly be interpreted as exhibiting nonlocality. Fortunately, we can put an upper bound of

$$E(X_N) \leq 3 + 5N^{-1/2+\epsilon} + 5\sqrt{3/2\pi}N^{-\epsilon} \exp(-N^{2\epsilon/6}) \quad (13)$$

for any small  $\epsilon > 0$ . Thus, for large enough  $N$ ,  $X_N$  is almost as good as  $Y_N$  at distinguishing quantum theory from local hidden variable models.

In the one-sided memory case, we can use the two-sided memory results to show that  $Y_N$  is unaffected by the presence of memory, and  $X_N$  is affected in a negligible way for sufficiently large  $N$ . Actually, we have not succeeded in finding a one-sided memory model for which  $E(X_N)$  or  $P(\hat{X}_N > \delta)$  are larger than the maximal values attainable by memoryless models for any  $N$ . We thus cannot exclude the possi-

bility that one-sided memory is of no use at all in helping LHV models come closer to reproducing quantum mechanics.

In the collective case,  $E(Y_N) \leq 3$ . However, we present a collective LHV model which has bigger fluctuations than are possible without memory. To have a reliable test of nonlocality, we need the fluctuations to become small as  $N$  gets large. We conjecture that this is indeed the case: however, the question marks in the table reflect the fact that we have no rigorous proof.

#### IV. CHSH-TYPE INEQUALITIES IN BELL'S NO MEMORY MODEL

We first revisit the derivation of the CHSH inequality in Bell's model, using techniques that will be useful for analyzing the different memory models.

We first recall how these quantities are interpreted in standard analyses, when the Bell pairs are measured sequentially and the memory loophole is neglected. Let  $Z_N$  be a binomially distributed variable with  $N$  trials, each of which has the two possible outcomes 0 and 1, with probability  $p \neq 0, 1$  of outcome 1 for each. The normal approximation to the binomial distribution gives us that

$$P(Z_N > pN + z\sqrt{Np[1-p]}) \rightarrow 1 - \mathcal{N}(z) \quad (14)$$

as  $N \rightarrow \infty$ , where

$$\mathcal{N}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp\left(-\frac{1}{2}y^2\right) dy \quad (15)$$

is the normal distribution function, which obeys

$$1 - \mathcal{N}(z) \approx \frac{1}{\sqrt{2\pi}} z^{-1} \exp\left(-\frac{1}{2}z^2\right). \quad (16)$$

For large  $N$ , and for  $z$  large compared to 1 and small compared to  $N^{1/2}$ , the errors in these approximations are small and can be rigorously bounded [11]. Below we consider  $N$  and  $z$  in these ranges and neglect the error terms, which make no essential difference to the discussion.

Now

$$Y_N = \frac{4}{N} \sum_{n=1}^N Y_N^n, \quad (17)$$

where

$$Y_N^n = \delta_c^n(A_1, B_1) + \delta_c^n(A_1, B_2) + \delta_c^n(A_2, B_1) + \delta_a^n(A_2, B_2). \quad (18)$$

Here  $\delta_c^n(A, B)$  is 1 if  $A$  and  $B$  are measured at the  $n$ th round and found to be the same, and 0 otherwise, and  $\delta_a^n(A, B)$  is 1 if  $A$  and  $B$  are measured at the  $n$ th round and found to be different, and 0 otherwise.

In a memoryless local hidden variable theory, the  $Y_N^n$  are independent random variables taking values 0 or 1. We have that

$$E(\delta_c^n(A_1, B_1)) = \frac{1}{4} p_c^n(A_1, B_1), \quad (19)$$

where  $p_c^n(A_1, B_1)$  is the probability that  $A_1 = B_1$  if  $(A_1, B_1)$  is measured at the  $n$ th trial, and similarly for the other three terms in Eq. (18). So, from Eq. (2) we have that

$$y_n = E(Y_N^n) = \frac{P_{CHSH}}{4} \leq \frac{3}{4}. \quad (20)$$

Clearly, for any  $N$  and any  $\delta > 0$ , the probability  $P(Y_N > 3 + \delta)$  is maximized when the  $Y_N^n$  are identically distributed, with  $y_n = 3/4$  for all  $n$ . For small  $\delta$  we have that

$$\begin{aligned} P(Y_N > 3 + \delta) &= P(NY_N/4 > 3N/4 + \delta N/4) \\ &\approx 1 - \mathcal{N}(\delta\sqrt{N}/\sqrt{3}) \approx \frac{1}{\sqrt{2\pi}} \frac{\sqrt{3}}{\delta\sqrt{N}} \\ &\quad \times \exp\left(-\frac{1}{6}\delta^2 N\right) \end{aligned} \quad (21)$$

for large  $N$ , which tends to zero fast as  $N \rightarrow \infty$ . A similar argument shows that quantum mechanics predicts that  $P(Y_N < 2 + \sqrt{2} - \delta)$  tends to zero fast. A long run of experiments can thus distinguish quantum mechanics and memoryless local hidden variables with near certainty.

Although the analysis of  $Y_N$  is simpler and arguably more natural, Bell experiments are traditionally interpreted via the quantity  $X_N$ . Since

$$\begin{aligned} E\left(\frac{\mathfrak{N}_c(A, B)}{\mathfrak{N}(A, B)}\right) &= \sum_{n=1}^N p(\mathfrak{N}(A, B) = n) \\ &\quad \times \frac{E(\mathfrak{N}_c(A, B) | \mathfrak{N}(A, B) = n)}{n} \\ &= \sum_{n=1}^N p(\mathfrak{N}(A, B) = n) \frac{n P_c(A, B)}{n} = P_c(A, B), \end{aligned} \quad (22)$$

and similarly  $E(\mathfrak{N}_a(A, B)/\mathfrak{N}(A, B)) = P_a(A, B)$ , Eqs. (2) and (10) imply that  $E(X_N) \leq 3$ . (Recall that we assume the  $n = 0$  terms in these sums have negligible probability.)

Moreover, since

$$P(\mathfrak{N}(A, B) < N/4[1 - \delta]) \approx \frac{\sqrt{3}}{\delta\sqrt{2\pi N}} \exp\left(-\frac{1}{6}\delta^2 N\right), \quad (23)$$

we have that

$$P\left(X_N > \frac{1}{1 - \delta} Y_N\right) \leq \frac{4\sqrt{3}}{\delta\sqrt{2\pi N}} \exp\left(-\frac{1}{6}\delta^2 N\right) \quad (24)$$

and

$$P\left(X_N > \frac{3 + \delta}{1 - \delta}\right) \leq \frac{5\sqrt{3}}{\delta\sqrt{2\pi N}} \exp\left(-\frac{1}{6}\delta^2 N\right). \quad (25)$$

Similarly, quantum mechanics predicts that  $P(X_N < 2 + \sqrt{2} - \delta)$  tends to zero fast. Thus, for large  $N$ ,  $X_N$  distinguishes

the predictions of quantum mechanics and memoryless local hidden variables almost as well as  $Y_N$  does.

## V. THE TWO-SIDED MEMORY LOOPHOLE

Now we consider the case where the LHV model for  $N$  trials is allowed to exploit the memory loophole, predicting results at each round of measurement, which may depend upon the previous measurements and outcomes on both sides.

Since Eqs. (19) and (20) still hold, we have that

$$E(Y_N) = \frac{4}{N} \sum_{n=1}^N E(Y_N^n) \leq \frac{4}{N} \sum_{n=1}^N \frac{3}{4} = 3. \quad (26)$$

Thus memory does not help increase  $E(Y_N)$ . We shall now show that it does not help the probability of a large fluctuation in  $Y_N$ . First, we note that  $Y_N$  is just (a constant times) the sum of  $Y_N^n$ , where  $Y_N^n$  is a random variable at the  $n$ th trial. Now,  $Y_N^n$  can only take values of 0 or 1. To maximize the probability of a large  $Y_N$ , we should try to maximize the probability of each  $Y_N^n$  being 1. This at first appears complicated, since with memory LHV models there will be correlations between  $P(Y_N^n = 1)$  for different  $n$ . The key is to note that, regardless of what happens in later rounds, for all LHV memory models,

$$P(Y_N^n = 1 | \text{events in trials } 1, \dots, n-1) \leq 3/4. \quad (27)$$

This is because, for any fixed set of events in the earlier rounds, the model in round  $n$  is just an LHV model, whose probabilities have been chosen with no prior knowledge of the measurements which will be performed in round  $n$ , and must therefore satisfy the CHSH inequality.

It follows that, for any  $N$  and any  $\delta > 0$ , the probability  $P(Y_N > 3 + \delta)$  is maximized when  $P(Y_N^n = 1) = 3/4$  for all  $n$ . But an LHV model can maximize the probability that  $Y_N^n = 1$ , for any  $n$ , by a strategy independent of the outcomes of the previous measurements, for instance, by predicting the outcome 1 for any measurement on either side. Since  $Y_N^n = 0$  or 1, any such strategy maximizes the probability  $P(Y_N > 3 + \delta)$ , and so Eq. (21) still holds even when the memory loophole is taken into account. The memory loophole does not alter the distinguishability of the predictions of quantum mechanics and local hidden variables, if  $Y_N$  is used as the correlation measure, since neither the maximal expectation nor the maximal variance of  $Y_N$  are increased by memory-dependent strategies.

Now let us turn to  $X_N$ . We know that if the particles are described by identical LHV models, then  $E(X_N) \leq 3$ . Also, even when the particles have memory, Eqs. (23)–(25) hold. Suppose we take  $\delta = N^{-1/2+\epsilon}$ , for some small  $\epsilon > 0$ , and let  $N$  be large enough so that  $(3 + \delta)/(1 - \delta) < 3 + 5\delta$ . Then from Eq. (25), since  $X_N$  is always bounded by 4, we have that

$$\begin{aligned} E(X_N) &\leq 4P(X_N > 3 + 5\delta) + (3 + 5\delta)[1 - P(X_N > 3 + 5\delta)] \\ &\leq 3 + 5N^{-1/2+\epsilon} + 5\sqrt{3/2\pi N}^{-\epsilon} \exp(-N^{2\epsilon}/6), \end{aligned} \quad (28)$$

so that  $E(X_N) - 3$  is bounded by a term that decays faster than  $N^{-1/2+\epsilon}$  for any  $\epsilon > 0$ . This means that no LHV model can produce  $E(X_N)$  much above 3 for large  $N$ ; it also means that the  $X_N$  remain efficient discriminators of quantum mechanics and local hidden variable theories even when the memory loophole is taken into account.

So far we have shown that the memory loophole makes no essential difference to Bell inequalities, so long as we use a large number of particles. We shall now show that if we only use a small number of particles, the two-sided memory loophole does indeed make a difference. We shall give a memory-dependent LHV model with  $E(X_N) > 3$ . To construct a simple example, we take a model which gives  $X_N = 3$  with certainty, and modify it a little so that the expectation increases above 3. We set  $N = 101$ . We can get  $X_{101} = 3$  with certainty, simply by outputting  $+1$  regardless of the observables measured. Our model is identical to this one, except for the case when, after 100 measurements, we have measured  $(A_1, B_1)$ ,  $(A_1, B_2)$ , and  $(A_2, B_1)$  33 times each, and  $(A_2, B_2)$  once. Our model is allowed memory, so that it can count how many times the various observables are measured, and thus tell when this is the case. In this (rather unlikely) case, this model will output  $+1$  on side  $A$  regardless of the measurement, and  $B_1 = +1$ , if measured, while  $B_2 = -1$ , if measured.

The two models will give identical values for  $X_{101}$  unless the above unusual state of affairs occurs after 100 rounds. Conditioned upon this event occurring, the first model still has an expectation of  $X_{101}$  equal to 3, whereas the second model has slightly more, almost  $25/8$ . Since the expectation of the present model is 3 in all other cases, this increases the unconditional expectation of the second model to very slightly greater than 3.

The intuition behind the modification is that if one term in  $X_N$  [e.g.,  $\mathfrak{N}_c(A_1, B_1)/\mathfrak{N}(A_1, B_1)$ ] has a small denominator compared to another term, then we will gain more by increasing the numerator in the term with the small denominator than in the term with the big denominator.

Now that we have this model with  $E(X_N) > 3$ , it is easy to see how to modify it to make a model that does better. The idea is to start trying to increase the numerator in the best places from the start. In each round, there are four possible pairs of observables which could be measured  $[(A_1, B_1), (A_1, B_2), \text{etc.}]$ . We can send a list that is guaranteed to give the correct sort of correlation or anticorrelation to at most three of the possible pairs, where we can choose which ones. So at each stage our model must choose one pair which, if measured, will give the wrong sort of correlation. After all the measurements are finished, the model would like to give the ‘‘incorrect’’ correlation to the pair of observables which has been measured most (since this term has the biggest denominator). There is no way for it to be sure of doing this, since it does not know at the start which pair will be measured most. So, our model simply guesses.

More precisely, the improved model is as follows. In the first round of measurements it gives the outcome  $+1$ , whatever is measured. From the second round it looks to see which pair, e.g.,  $(A_1, B_2)$ , has been measured most, and arranges that if that pair is measured in the next round, the

correlations will be incorrect, whereas if any other pair is measured in the next round the correlations will be correct. It is easy to see that this model produces  $E(X_N) > 3$  for all  $N$  large enough so that there is a negligible probability of one of the four observable pairs not being measured. Of course, our earlier bounds imply that  $E(X_N) \rightarrow 3$  as  $N \rightarrow \infty$ . We conjecture that the model produces the maximum value of  $E(X_N)$  attainable by a local hidden variable theory with two-sided memory.

## VI. THE ONE-SIDED MEMORY LOOPHOLE

We comment briefly on the case of the one-sided memory loophole, represented by a model of the form (5). We do not know whether such models can increase the value of  $E(X_N)$  above 3, or come any closer to simulating quantum theory than memoryless LHV models. Note, however, that one-sided memory models are a restricted class of the two-sided memory models, and thus all the upper bounds proven for two-sided models still apply. In particular,  $E(Y_N) \leq 3$ , and Eq. (21) still holds, i.e.,  $P(Y_N > 3 + \delta) \approx (1/\sqrt{2\pi}) \times (\sqrt{3}/\delta\sqrt{N}) \exp(-\frac{1}{6}\delta^2 N)$ . These are in fact tight bounds, since they can be obtained without any memory.

The two-sided bounds also apply for  $X_N$ . However, we do not know whether they are tight: it may be that one-sided memory LHV models are no more powerful than memoryless LHV models.

## VII. SIMULTANEOUS MEASUREMENT LOOPHOLE

Although we have seen that the memory loophole gives LHV models some small wiggle room, it makes little essential difference. Both  $Y_N$  and  $X_N$  remain efficient discriminators between the predictions of quantum mechanics and of LHV models in the presence of the memory loophole. One might conjecture that this is also true of the simultaneous measurement loophole, and that this can be shown by essentially the same argument.

However, things are more complicated. It is true that  $E(Y_N) \leq 3$ , since Eq. (26) still holds, following the same reasoning as before. However, the fluctuations cannot be dealt with so easily. Our arguments to date have relied on the fact that  $P(Y_N^n = 1) \leq 3/4$  holds true in any LHV model, not just *a priori*, but after conditioning on events up to round  $(n-1)$ , even when the memory loophole allows the behavior of the round  $n$  LHVs to depend on the earlier results. In particular, we have that

$$P(Y_N^n = 1 \mid Y_N^{n-1} = i_{n-1}, \dots, Y_1^n = i_1) \leq 3/4 \quad (29)$$

for any values of  $i_1, \dots, i_{n-1}$ , in any memory-dependent LHV model.

However, the derivation of Eq. (29) relies on the fact that results from earlier rounds are necessarily uncorrelated with measurement choices from later rounds. This need not be true when the simultaneous measurement loophole can be exploited, as the following simple example illustrates.

Take  $N = 2$ , and consider simultaneously measured local hidden variables with the following outcome rules: on side

A, the outcomes are (1,1) unless the operators measured are  $(A^1, A^2)$ , when the outcomes are (1,0); on side B, the outcomes are (1,1) unless the operators measured are  $(B^2, B^1)$ , when the outcomes are (0,1).

Here the outcomes and operators are ordered so that, for example, an outcome  $(i, j)$  means that  $i$  was obtained on the relevant particle from the first pair, and  $j$  was obtained on the relevant particle from the second pair. The pairs themselves are ordered by some convention: it does not matter which, so long as the ordering is consistent on each side.

It is easy to verify that, in this model,

$$P(Y_2^1=1 \text{ and } Y_2^2=1)=10/16, \quad (30)$$

whereas Eq. (29) would imply

$$P(Y_2^1=1 \text{ and } Y_2^2=1)\leq 9/16. \quad (31)$$

In other words, the simultaneous memory loophole allows an LHV model to increase the probability of getting a larger than expected value for  $Y_2$ , beyond that attainable by any model in which the  $Y_N^n$  are independent random variables. The arguments of the preceding sections thus no longer apply.

We conjecture, nonetheless, that the predictions of quantum mechanics and of local hidden variables using the simultaneous measurement loophole can be discriminated by  $Y_N$  for large  $N$ . If so, then in theory, the detector efficiency loophole could be countered by setting up an experiment in which a single pair of photons simulates  $N$  spin singlet states: i.e., many degrees of freedom of a single pair of photons are entangled, so that the joint state is isomorphic to the state of  $N$  singlets. One could then choose random measurements on each photon, which simulate independent measurements on individual photons in the  $N$  singlets. Ignoring (admittedly somewhat unrealistically) losses in the beam splitters used to set up the measurements, this means that results for all  $N$  simulated singlets are obtained whenever the detectors on both sides fire. If both detectors are of efficiency  $f$ , this will happen with probability  $f^2$ —a gain of  $f^{2N-2}$  over the probability of obtaining a full set of results if  $N$  pairs of photons were separately measured. Choosing small  $\epsilon$ , and taking  $N$  such that  $P(Y_N > 3 + \epsilon) \ll f^2$  for any collective local hidden variable model would allow the hypothesis of collective local hidden variables to be refuted in a single successful experiment (which will take approximately  $f^{-2}$  attempts).

A strategy for combatting detector efficiency which uses the same basic idea of working with a highly entangled state of two photons, but is conceptually rather different, has been proposed by Massar [12].

## VIII. CONCLUSION

We have seen that in the analysis of Bell-type experiments, one ought to allow for the possibility that the particles have memory, in the sense that outcomes of measurements on the  $n$ th pair of particles depend on both measurement choices and outcomes for the 1st,  $\dots$ ,  $(n-1)$ th pairs. The standard form for local hidden variable models, originally due to Bell and summarized in Eq. (1), does not allow for this possibility, so a new analysis is needed. We have distinguished one-sided and two-sided versions of this loophole and shown that in the two-sided case, a systematic violation of a Bell-type inequality can be obtained. In the case of the CHSH inequality, however, we have derived an upper bound on the probability of large deviations and thereby shown that the expected violation tends to zero as the number of particle pairs tested becomes large. Thus the CHSH inequality is robust against the memory loophole and the corresponding experimental tests remain good discriminators between quantum mechanics and local hidden variables—there is no need to design improved experiments in which more (or even all) measurements are spacelike separated from one another.

We have also shown that if the analysis is performed in terms of the quantities  $Y_N$ , rather than  $X_N$ , then the memory models give no advantage over standard, memoryless, local hidden variables. Finally, we have considered a related loophole, the simultaneous measurement loophole, which would arise if Alice and Bob each performed all their measurements simultaneously, thus allowing for collective local hidden variables. We have seen that in this case, the probability of a significant deviation above the CHSH bound can be larger than would be allowed for a standard local hidden variable model. However, we suspect that this extra freedom is small, in the sense that the predictions of collective local hidden variables can be distinguished from those of quantum mechanics for a large enough number of particle pairs.

*Note added.* After this work was completed, we became aware of independent work by Gill [13], in which similar bounds on the probabilities of simulating quantum-mechanical results via memory loophole local hidden variable models are presented. The existence of the memory loophole was independently noticed by Accardi and Regoli [14], whose speculation that it might allow local hidden variables to simulate quantum mechanics is refuted by Gill's (and our) analyses.

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